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EXISTENTIALLY COMPLETE LATTICE-ORDERED GROUPS[†]

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ABSTRACT

We investigate existentially complete lattice-ordered groups in this paper. In particular, we list some of their algebraic properties and show that there are continuum many countable pairwise non-elementarily equivalent such lattice-ordered groups. In particular, existentially complete lattice-ordered groups give rise to a new class of simple groups.

Our main aim in this paper is to show that existentially complete lattice-ordered groups are in some sense as complicated as existentially complete groups. Indeed, they yield an abundance of new simple lattice-ordered groups. Specifically, we prove:

THEOREM (Algebra). *Existentially complete lattice-ordered groups are divisible, simple (even as groups), not laterally complete, contain no basic or special elements, contain no finite maximal pairwise disjoint set of elements, and are neither finitely generated nor finitely related. Any two positive elements of an existentially complete lattice-ordered group are conjugate, and any element is a commutator. Moreover, algebraically closed lattice-ordered groups are existentially complete or trivial.*

THEOREM (Logic). *The theory of lattice-ordered groups has no model companion. Indeed, there are* $2^{\aleph_0}$ *pairwise non-elementarily equivalent countable existentially complete lattice-ordered groups. The finite forcing companion is complete and hyperarithmetic (as complicated as first order number theory); the infinite forcing companion is complete and* Δ_1^2 *(as complicated as second order number theory).*

[†] This paper is dedicated to the memory of Abraham Robinson. Without his pioneer work in model-theoretic forcing, none of this research would have been posssible.

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Lattice-ordered groups are torsion-free (see [8], lemma 1 or [3]). Consequently, existentially complete lattice-ordered groups are not existentially complete qua groups (existentially complete groups have infinitely many elements of all finite orders). What is much worse is that the amalgamation property (present for groups and for lattices) fails for lattice-ordered groups [16]. This means that our methods are radically different from those of Belagradek [2], Macintyre [12] and Ziegler [20] for groups (all use free products with amalgamation heavily). In the absence of the amalgamation property, we are forced to look at some kind of universal lattice-ordered groups. Ordered permutation groups serve the purpose and we use them very heavily. We admit the crassness of this approach. Whereas the free product with amalgamation allows one to enlarge a group just enough to capture some extra property (e.g., divisibility), passing to permutation groups throws in a great deal of extra junk that has to be sifted out in trying to close in on the existentially complete group being considered. Our defenses are that it works and without the amalgamation property, what else can one do? What is quite surprising is that so much can be obtained, if rather clumsily, by this approach (see, in particular, the proof of crucial Lemma 16). We advise the reader to get an overall picture of the remarks and then come back to some of the more technical proofs. Despite the niceness of the general picture, some proofs require not only a knowledge of some obscure results on ordered permutation groups, but also a knowledge of some proofs in that subject. The readers who wish to follow the gory details will need to have a copy of [6] handy — they'll use it!

Since we have tried to write this paper so that algebraists and model-theorists alike will be able to understand most of it, there will be portions that each can (and should!) easily skip over.

The results in this paper were first announced at the "Ordered Groups Conference" in Boise, Idaho, in October 1978. Statements of the theorems appear in the *Proceedings* of that conference [7].

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forcing to algebra. All of section 3 was directed by that awareness as was the crucial Lemma 16.

0. Background information and notation

0.1. *Algebra*

A *lattice-ordered group* (*l-group*) is a group endowed with two extra binary operations $\vee$ and $\wedge$ making it a lattice such that the group operation distributes over the lattice operations ($x(y \vee z) = xy \vee xz$, $(y \vee z)x = yx \vee zx$, and dually for $\wedge$). l-groups are torsion-free (see [8] lemma 1) and the lattice is distributive [3]. Moreover, $f \wedge g = (f^{-1} \vee g^{-1})^{-1}$ (so $\wedge$ is unnecessary) [3]. Under the usual ordering, the additive groups of integers $\mathbf{Z}$ and reals $\mathbf{R}$ are l-groups.

Write $f \leqq g$ if $f \vee g = g$. If G is an l-group, $g \in G$ is said to be *positive* if $g > e$, where e is the group identity. We use G^+ for $\{g \in G : g \text{ is positive}\}$. If $\{G_i : i \in I\}$ is a family of l-groups, $\Pi_{i \in I} G_i$ is an l-group under pointwise order.

If G is an l-group, $f, g \in G$ are said to be *disjoint* if $f \wedge g = e$, and $g \in G$ is said to be *basic* if $g > e$ and $\{f \in G^+ : f < g\}$ is totally ordered (with respect to $\leqq$).

An l-group G is said to be *laterally complete* if every set of pairwise disjoint elements of G has a supremum in G.

Let G be an l-group. H is an *l-subgroup* of G (written $H \leqq G$) if H is a subgroup and a sublattice of G. Observe that if $\{e\} \neq H \leqq G$, then H contains a positive element (if $g \in H$ and $g \neq e$, $g \vee e \in H$ and is positive unless $g \leqq e$, in which case $e \neq g^{-1} \in H$ and is positive). If K is an l-group and $\phi : G \to K$ is a group and lattice homomorphism, then ϕ is called a *l-homomorphism*. If, in addition, ϕ is one-to-one, it is called an *l-embedding*; in this case, we will often identify G with $G\phi$ and write $G \leqq K$ ($G\phi$ is an l-subgroup of K if ϕ is an l-homomorphism). The kernels of l-homomorphisms of G into l-groups range over all the normal convex l-subgroups of G. An l-group whose only normal convex l-subgroups are the obvious two is said to be *l-simple*. Note that l-groups that are simple as groups are l-simple.

Let Ω be a totally ordered set. The group $A(\Omega) = \mathrm{Aut}(\Omega, \leqq)$ of all order-preserving permutations of Ω is an l-group under the ordering $f \leqq g$ if and only if $\alpha f \leqq \alpha g$ for all $\alpha \in \Omega$. Define the *support* of $g \in A(\Omega)$ to be $\mathrm{supp}(g) = \{\alpha \in \Omega : \alpha \neq \alpha g\}$. If $\alpha \in \mathrm{supp}(g)$ then the *supporting interval* of g containing α, denoted by $\Delta(\alpha, g)$, is the convexification of $\{\alpha g^n : n \in \mathbf{Z}\}$. *g depressed off* $\Delta(\alpha, g)$ is that $f \in A(\Omega)$ defined by: $\beta f = \beta g$ for $\beta \in \Delta(\alpha, g)$ and $\beta f = \beta$ otherwise. If G is an l-subgroup G of $A(\Omega)$ we write (G, Ω) when it is important to know what underlying set is being permuted. (G,Ω) is said to be *depressible* if it contains, together with any g, all of its depressions off supporting intervals. (G,Ω) is *o-2-transitive* if, whenever $\alpha,\beta,\sigma,\tau\in\Omega$, $\alpha<\beta$ and $\sigma<\tau$, there exists $g\in G$ taking α to σ and β to τ. o-2-transitivity implies o-n-transitivity for every $n\in\mathbf{Z}^+$, [6, lemma 1.8.12]. We say Ω is *o-2-homogeneous* if $(A(\Omega),\Omega)$ is o-2-transitive. In this case $A(\Omega)$ is divisible. Every l-group can be l-embedded in $A(\Omega)$ [6, theorem 1.8.2]. In fact, Ω can be chosen o-2-homogeneous [6, chapter 8] or [16, lemma 4.3]. This plays a key role in the crucial Lemma 16.

An important l-subgroup of $A(\Omega)$ is the group $B(\Omega)$ of all $g\in A(\Omega)$ having bounded support. Note that $A(\Omega)$ is laterally complete but $B(\Omega)$ is not if $B(\Omega)\neq\{e\}$.

If $\Omega\subseteq\Omega^*$ then (G,Ω) is said to be *expanded* to $(G\theta,\Omega^*)$ if $\theta: G\to A(\Omega^*)$ is an l-embedding for which $\alpha g=\alpha(g\theta)$ for all $\alpha\in\Omega$. If $\bar\Omega$ denotes the Dedekind completion of Ω then $A(\Omega)$ can be expanded naturally to $(A(\Omega),\bar\Omega)$ by defining $\bar\alpha g=\sup\{\alpha g:\alpha\in\Omega \text{ and } \alpha<\bar\alpha\}$. We pass freely from Ω to $\bar\Omega$ as the need arises. We will need one more fact for Lemma 16: If $e<g_1$, $g_2\in A(\Omega)$, $\alpha,\beta\in\Omega$, $\alpha<\alpha g_1$, $\beta<\beta g_2$, then there is $\Omega\subseteq\Omega^*$ and an expansion to $A(\Omega^*)$, and there is $h\in A(\Omega^*)$ such that $\alpha h=\beta$ and $h^{-1}g_1h=g_2$ (see [16, proof of corollary 5.2] or [6, proof of theorem 8.3.2]). Consequently every l-group is embeddable in an l-group in which any two positive elements are conjugate. The proofs of this result and the others mentioned from [16] use α-sets (η_α-sets of cardinality $\aleph_\alpha$), whose existence is equivalent to the Generalized Continuum Hypothesis. For those who, like the first author, expect the *dies irae* to come momentarily when G.C.H. is used, there is a way of avoiding it using "order small" sets [19].

If H is an l-group and (G,Ω) is transitive, $H\,\mathrm{Wr}(G,\Omega)$ will denote the semidirect product of $\Pi_{\alpha\in\Omega}H$ by G in which

$$(\{h_\alpha\},g)\cdot(\{f_\alpha\},g')=(\{k_\alpha\},gg'),$$

where $k_\alpha=h_\alpha f_{\alpha g}$. $(\{h_\alpha\},g)\geqq e$ if and only if $g\geqq e$ and $h_\alpha\geqq e$ for all $\alpha\in\Omega$ such that $\alpha g=\alpha$.

Any other terms used in this paper will not affect the major results. The interested reader can find the definitions in [3] or [6].

0.2. *Logic*

Since we are using $\vee$ and $\wedge$ for the lattice operations, we will use *or* and $\mathbb{V}$ for disjunction, and & and $\mathbb{A}$ for conjunction.

Recall that if T is a theory and $\mathfrak{A}$ is a submodel of a model of T, then $\mathfrak{A}$ is said to be *existentially complete* in T, or T-existentially complete, if any existential sentence of $\mathfrak{L}(\mathfrak{A})$ (the language $\mathfrak{L}$ augmented by constants naming the elements of $\mathfrak{A}$) which is true in a model of T containing $\mathfrak{A}$, is already true in $\mathfrak{A}$. If instead of considering all existential sentences of $\mathfrak{L}(\mathfrak{A})$, we restrict to those obtained by existentially quantifying over conjunctions of atomic sentences, we call the corresponding structures *algebraically closed* in T. In the presence of the group axioms, algebraically closed corresponds to having a solution to $w_0(\boldsymbol{x}, \boldsymbol{a}) = e, \cdots, w_{n-1}(\boldsymbol{x}, \boldsymbol{a}) = e$ in $\mathfrak{A}$ whenever there is a solution in some group $\mathfrak{B} \supseteq \mathfrak{A}$, and existentially complete allows also $w_n(\boldsymbol{x}, \boldsymbol{a}) \neq e, \cdots, w_m(\boldsymbol{x}, \boldsymbol{a}) \neq e$ $(\boldsymbol{a} \in \mathfrak{A})$. For T the theory of fields, the concepts coincide ($w \neq 0$ can be replaced by solving $wy - 1 = 0$) and yield the standard algebraically closed fields. For abelian torsion-free groups, the concepts coincide and yield the class of divisible torsion-free abelian groups (except that $\{0\}$ is algebraically closed but not existentially complete).

A theory T has a *model companion* theory T^c if and only if:

(1) every model of T is contained in a model of T^c and conversely, and

(2) T^c is model-complete; i.e., every model of T^c is existentially complete in T^c.

If it exists, the model companion is unique. It is easy to see that every (infinite) model of T is contained in a T-existentially complete model (of the same cardinality, if $\mathfrak{L}$ is countable). Consequently, if the T-existentially complete structures form an elementary class (i.e., there is a set of sentences S whose models are precisely the T-existentially complete structures), then this class yields the model companion; if it isn't an elementary class, the model companion doesn't exist. In the pressence of the amalgamation property, the model companion is called the *model completion*.

Special cases of existentially complete structures are the finitely generic and infinitely generic ones. The infinitely generic ones are large in the sense that if $\mathfrak{A}$ is a model of T, there is an infinitely generic structure $\mathfrak{B}$ containing $\mathfrak{A}$ as a submodel. If T enjoys the joint embedding property (the theory of lattice-ordered groups does since the direct sum of two lattice-ordered groups is a lattice-ordered group), any two infinitely generic structures satisfy the same first order sentences. This is all we'll need about them. The finitely generic structures on the other hand are small in the sense that if $\mathfrak{A} \subseteq \mathfrak{B}$, $\mathfrak{A}$ is existentially complete in T and $\mathfrak{B}$ is finitely generic, then $\mathfrak{A}$ is finitely generic. Since our language is countable and our theory enjoys the joint embedding property, we have that the finitely generic structures are among the T-existentially complete structures that satisfy the complete theory T^f (T^f is the set of sentences ϕ of $\mathfrak{L}$ such that

$\varnothing \Vdash \neg\neg \phi$. Let C be a countable set of constants not occurring in $\mathfrak{L}$. A finite set p of basic sentences (atomic or negated atomic) of $\mathfrak{L}(C)$ is called a *condition* if $T \cup p$ is consistent. If p is a condition and ϕ is a sentence of $\mathfrak{L}(C)$, $p \Vdash \phi$ (p *forces* ϕ) is defined inductively: for ϕ atomic, if $\phi \in p$; for &, or, $\exists$ do the obvious; but $p \Vdash \neg \psi$ if no condition $q \supseteq p$ forces ψ).

A final comment on notation: we shall use the same symbol to denote an elementary class and the first order theory of that class.

If this review has been inadequate, see the appendix (The Lazy Algebraist's Guide to Model-Theoretic Forcing) to our article [7], G. Cherlin's book [4], or J. Hirschfeld and W. H. Wheeler's book [10]. This last is very complete and thorough.

1. Some easy first order algebraic properties

LEMMA 1. *Every existentially complete l-group is divisible.*

PROOF. Follows trivially from the fact that every l-group is l-embeddable in a divisible l-group [6, theorem 7.2.7].

LEMMA 2. *Any two positive elements of an existentially complete l-group are conjugate. Consequently existentially complete l-groups are l-simple.*

PROOF. This is an $\forall\exists$ sentence which by 0.1 holds in an l-extension, so holds in the existentially complete l-group.

LEMMA 3. *Every element of an existentially complete l-group is a commutator. Indeed, every element is conjugate to its square.*

PROOF. Let G be existentially complete. l-embed G in $A(\Omega)$ for some doubly homogeneous totally ordered set Ω, and identify G with its image. Let $g \in G$. Then g, g^2 have the same supporting intervals and on any interval of support, both are positive or both are negative. Hence $\mathcal{A}(\Omega) \models \exists x\,(x^{-1}gx = g^2)$ (see [6], page 159). Thus $x^{-1}gx = g^2$ for some $x \in G$; i.e., $g = [g, x]$.

LEMMA 4. *No element of an existentially complete l-group is basic or special.*

PROOF. Let G be existentially complete and $e < g \in G$. l-embed G in $G \oplus G$ via $f \mapsto (f, f)$. $G \oplus G \models \exists x_0 \exists x_1 (x_0 \wedge x_1 = e \;\&\; x_0 \neq e \;\&\; x_1 \neq e \;\&\; x_0 \vee x_1 = (g, g))$ so $G \models \exists x_0 \exists x_1 (x_0 \wedge x_1 = e \;\&\; x_0 \neq e \;\&\; x_1 \neq e \;\&\; x_0 \vee x_1 = g)$; i.e., g is not basic. If g were special, then the same would be true of every positive element of G (by Lemma 2). Hence G is an o-group, so g is basic, a contradiction.

LEMMA 5. *No finite set of elements of an existentially complete l-group can be a maximal pairwise disjoint set.*

PROOF. Let $g_0, \cdots, g_{n-1}$ be a set of pairwise disjoint elements of an existentially complete l-group G. $G \oplus \mathbf{Z} \models \exists x \, (x \neq e \;\&\; \bigwedge_{i=0}^{n-1} (x \wedge g_i = e))$, so G contains a positive element disjoint from $g_0, \cdots, g_{n-1}$.

LEMMA 6. *If G is an existentially complete l-group, $e < g \in G$ and $n \in \mathbf{Z}^+$, there exists $x \in G$ such that $\{x^{-i} g x^i : 0 \leqq i \leqq n\}$ are pairwise disjoint.*

PROOF. l-embed G in $G \operatorname{Wr} \mathbf{Z}$ via $f \mapsto (\{f_n\}, 0)$ where

$$f_n = \begin{cases} f & \text{if } n = 0, \\ e & \text{if } n \neq 0. \end{cases}$$

Then $G \operatorname{Wr} \mathbf{Z} \models (\exists x) \bigwedge_{0 \leqq i < j \leqq n} (x^{-i} g x^i \wedge x^{-j} g x^j = e)$ (let $x = (\{e\}, 1)$), and thus the same holds in G.

Lemma 6 compares nicely with the result for nilpotent groups: If G is an existentially complete nilpotent group and $n \in \mathbf{Z}^+$, there is $g \in G$ such that, modulo the center of G, g has order greater than n. This idea turned out to be most useful in distinguishing between finitely generic and infinitely generic nilpotent groups (see [17]). Lemma 6 should therefore be borne in mind for later use.

LEMMA 7. *Every existentially complete l-group is simple (as a group).*

PROOF. Let G be existentially complete and $f, g \in G$ with $g \neq e$. l-embed G in $B(\Omega)$ for some o-2 homogeneous Ω. By Lemma 3 and [6], theorem 3.6.5, there are $x_0, x_1, x_2, x_3 \in B(\Omega)$ such that

$$f = x_0^{-1} g^{-1} x_0 \cdot x_1^{-1} g x_1 \cdot x_2^{-1} g^{-1} x_2 \cdot x_3^{-1} g x_3.$$

Hence such x_0, x_1, x_2, x_3 exist in G, so the normal subgroup of G generated by g is G.

Let $C(X) = \{g \in G : xg = gx \text{ for all } x \in X\}$.

LEMMA 8. *Existentially complete l-groups are neither finitely generated nor finitely related.*

PROOF. Let G be existentially complete and let $g_0, \cdots, g_{n-1} \in G$. $G \oplus \mathbf{Z} \models \exists x \, (x \neq e \;\&\; \bigwedge_{i=0}^{n-1} x g_i = g_i x)$ so the same sentence holds in G. Thus $C(\{g_0, \cdots, g_{n-1}\}) \neq \{e\}$. But since G is clearly non-abelian, $Z(G) = \{e\}$ by

Lemma 7. Hence $\{g_0, \cdots, g_{n-1}\}$ does not generate G. If G were finitely related, therefore, infinitely many of the generators of G could not occur in any of the finite number of defining relations. Hence they generate a free l-group F which is a free factor of G; say, G is the l-group free product of F and H. Let $g \neq e$ be one of the generators of F. By Lemma 3, there is $x \in G$ such that $x^{-1}gx = g^2$. Let $L = H \oplus \mathbf{Z}$ and map G onto L via ϕ where $\phi\,|_H =$ identity, $g\phi = (e, 1)$ and $f\phi = (e, 0)$ for all other generators f of F. This is an l-homomorphism so $(x\phi)^{-1}(g\phi)(x\phi) = (g\phi)^2$. But $g\phi \in Z(L)$ so $(e, 1) = g\phi = (g\phi)^2 = (e, 2)$, a contradiction.

B. H. Neumann [15] proved the infinite relatedness for groups by simply observing $Z(F) = \{e\}$ if F is a free group on infinitely many generators. Ashok Kumar has recently shown that the same is true for free l-groups and a proof could be given along Neumann's lines here, too. However, the one given has the virtue of being self-contained.

We now complete the proof of the Algebra Theorem stated in the introduction (with the exception of the lateral incompleteness clause, which is much harder) by proving:

THEOREM 9. *Every algebraically closed l-group is trivial or existentially complete.*

This follows at once from a result of P. Bacsich (see [5] theorem 6.5, or [7], appendix theorem 3) since every l-group can be l-embedded in an l-simple l-group ($B(\Omega)$ is l-simple if Ω is 0-2 homogeneous). We give a more algebraic proof patterned on the lines of B. H. Neumann's for groups [14] — it will help prepare the reader for the next section. Theorem 9 also holds for lattices [11]. Both Jónsson's[†] and Neumann's proofs use the amalgamation property which, alas (as can't be stressed too often), l-groups fail to enjoy.

To prove Theorem 9, we require two lemmas:

LEMMA 10. *Let G be an l-group, $e < g_0 \in G$ and $g \in G$. Suppose $u_i(x, g)$ and $v_j(x, g)$ are l-group words and that there is $H \geqq G$ satisfying, for some $h \in H$,*

(1) $u_i(h, g) = e$ $(1 \leqq i \leqq N)$ *and*

(2) $v_j(h, g) \neq e$ $(1 \leqq j \leqq M)$.

Then there is $K \geqq H$ in which (1)–(5) *are satisfied where*

(3) $z_j^2 = v_j(h, g)^{-1} z_j v_j(h, g)$ $(1 \leqq j \leqq M)$,

[†] We are not aware of any proof in the literature. B. Jónsson kindly provided us with a solution in reply to our query if the analogue of Theorem 9 held for lattices.

(4) $y_j^{-1}z_jy_j = z$ $(1 \leqq j \leqq M)$,
(5) $y^{-1}zg_0y = z$.

PROOF. Assume the hypotheses and write v_j for $v_j(\boldsymbol{h}, \boldsymbol{g})$ $(1 \leqq j \leqq M)$. l-embed H in $\mathcal{A}(\Omega)$ for some o-2 homogeneous Ω. Define $f_j \in \mathcal{A}(\Omega)^+$ by: $\alpha f_j = \alpha$ if $\alpha v_j = \alpha$. If $\alpha < \alpha v_j$, choose $\alpha, \tau \in (\alpha, \alpha v_j)$ so that $\sigma < \tau$. There exists $a_j \in \mathcal{A}(\Omega)^+$ such that $\sigma a_j = \tau$ and $\beta a_j = \beta$ if $\beta \notin (\alpha, \alpha v_j)$. Let $f_{j,0} = a_j$ and define $f_{j,n}$ inductively $(n \in \mathbf{Z}^+)$ to satisfy $f_{j,n+1}^2 = v_j^{-1}f_{j,n}v_j$ $(n = 0, 1, 2, \cdots)$. (Such $f_{j,n+1}$ exists since $\mathcal{A}(\Omega)$ is divisible.) Observe $\operatorname{supp}(f_{j,n}) \subseteq [\alpha v_j^n, \alpha v_j^{n+1}]$. For $n \leqq 0$, let $f_{j,n-1} = v_j f_{j,n}^2 v_j^{-1}$. Again $\operatorname{supp}(f_{j,n}) \subseteq [\alpha v_j^n, \alpha v_j^{n+1}]$. For $\beta \in \Delta(\alpha, v_j)$, let $n \in \mathbf{Z}$ be such that $\beta \in [\alpha v_j^n, \alpha v_j^{n+1}]$. Define $\beta f_j = \beta f_{j,n}$. Similarly, if $\alpha v_j < \alpha$. Now if $\beta \in \Delta(\alpha, v_j)$,

$$\beta v_j^{-1} f_j v_j = \beta v_j^{-1} f_{j,n-1} v_j = \beta f_{j,n}^2 = \beta f_j^2$$

(where $\beta \in [\alpha v_j^n, \alpha v_j^{n+1}]$). Hence $v_j^{-1}f_jv_j = f_j^2$ is satisfied in $\mathcal{A}(\Omega)$ with $f_j > e$ $(1 \leqq j \leqq M)$. Now $\mathcal{A}(\Omega)$ can be l-embedded in an l-group K in which any two positive elements are conjugate. Let $e < f \in K$. Then f_j and f are conjugate $(1 \leqq j \leqq M)$. Also, fg_0 and f, being both positive are conjugate, so (1)–(5) hold in K.

LEMMA 11. *Let $e < g_0 \in G$ and let l-group $L \geqq G$ satisfy* (1), (3), (4) *and* (5); *then any solution also satisfies* (2).

PROOF. Obvious.

Now we prove Theorem 9. Assume $\{e\} \neq G$ is algebraically closed and assume $u_i(\boldsymbol{x}, \boldsymbol{g}) = e$ $(1 \leqq i \leqq N)$, $v_i(\boldsymbol{x}, \boldsymbol{g}) \neq e$ $(1 \leqq i \leqq M)$ have a solution in some $H \geqq G$. By Lemma 10, there is l-group $K \geqq H$ satisfying (1)–(5). But $G \leqq K$ and G is algebraically closed. Hence there is a solution of (1), (3), (4) and (5) in G. By Lemma 11, G also satisfies (2), so G satisfies (1) and (2). Consequently, G is existentially complete.

We complete this section with three observations:

LEMMA 12. *If G is existentially complete, the only variety of l-groups containing G is the variety of all l-groups.*

PROOF. Let $\mathfrak{V}$ be a proper variety of l-groups. Let H be an l-group with $H \notin \mathfrak{V}$. Then there is a law of $\mathfrak{V}$ which fails in H, i.e., $H \models \exists \boldsymbol{x}(w(\boldsymbol{x}) \neq e)$ for some word w. Hence $G \oplus H \models \exists \boldsymbol{x}(w(\boldsymbol{x}) \neq e)$, so $G \models \exists \boldsymbol{x}(w(\boldsymbol{x}) \neq e)$; i.e., $G \notin \mathfrak{V}$.

LEMMA 13. *There are $2^{\aleph_0}$ pairwise non-isomorphic countable existentially complete l-groups.*

The proof is that of proposition 9 of [8].

Since existentially complete l-groups are l-simple, they have a transitive representation. That is, if G is existentially complete, there is a totally ordered set Ω such that G is l-isomorphic to a transitive l-subgroup of $\mathcal{A}(\Omega)$. By Lemma 2, this transitive representation (G,Ω) is not depressible. If (G,Ω) were o-primitive, then it would not be regular since G is non-abelian. Also, it cannot be periodic since periodic o-primitive l-groups contain positive elements which fix no points of $\bar{\Omega}$ as well as ones which fix points of Ω (and clearly none of the first kind can be conjugate to any of the second kind). Hence, by [6] theorem 3.3.12, if (G,Ω) were o-primitive, it would be o-2 transitive. There are two classes of o-2 transitive l-permutation groups: those that have elements of bounded support and those that don't. Those in this latter class are known as *pathological* and live up to their name.

LEMMA 14. *If G is an existentially complete l-group and (G,Ω) is transitive o-primitive, then it is pathologically o-2 transitive.*

PROOF. Let $e < g \in G$. l-embed G in $\Pi_{n\in\mathbf{Z}} G$ diagonally and hence l-embed G in $H = (G,\Omega)\mathrm{Wr}(\mathbf{Z},\mathbf{Z})$. Let $e \leqq g_1 \in G$. Now

$$H \models (\exists x)(x > e \ \&\ g \wedge x^{-1}g_1^{-1}gg_1x \neq e),$$

and hence $G \models (\exists x)(x > e \ \&\ g \wedge x^{-1}g_1^{-1}gg_1x \neq e)$. Thus by [6] lemma 3.6.1, g does not have bounded support. Since this is for all $g > e$, (G,Ω) is pathologically o-2 transitive.

If any existentially complete l-group has an o-primitive transitive representation, it yields a previously unknown simple pathologically o-2 transitive l-group. If any has no o-primitive transitive representation, we would have our first example of an l-simple l-group that has no o-primitive transitive representation and yet is not an o-group (sic!). So in either case — and we don't know which (possibly both) — we have some very new, rich, exotic l-groups. Would we knew them better!

2. Model companions — finite and infinite forcing companions

The development is similar to that for many other theories: there is a non-first-order property of l-groups which is captured by a first order sentence when we restrict to existentially complete l-groups. The ramifications of this fact permeate the rest of the paper. First we note an important property of commuting order-preserving permutations:

LEMMA 15. *Suppose $e < f$, $g \in A(T)$ and Γ and Δ are supporting intervals of f and g respectively. If $fg = gf$, exactly one of the following occurs:*
(i) $\Gamma = \Delta$,
(ii) $\Gamma \cap \Delta = \varnothing$,
(iii) $\Delta \subseteq (\beta, \beta f)$ *for some* $\beta \in \bar{\Gamma}$, *or*
(iv) $\Gamma \subseteq (\delta, \delta g)$ *for some* $\delta \in \bar{\Delta}$.

PROOF. If Γ and Δ are neither disjoint nor equal then one of $\inf(\Gamma) \in \bar{\Delta}$, $\sup(\Gamma) \in \bar{\Delta}$, $\sup(\Delta) \in \bar{\Gamma}$, or $\inf(\Delta) \in \bar{\Gamma}$ holds. In the first case let $\delta = \inf(\Gamma)$. Then $\delta f = \delta$ and $\delta g f = \delta f g = \delta g$, whence $\Gamma \subseteq (\delta, \delta g)$. The other cases are treated similarly.

Let $\langle g \rangle$ denote the subgroup generated by g. Note that "$f \in \langle g \rangle$" is an $\mathfrak{L}_{\omega_1 \omega}$ formula in all l-groups ($\bigvee\!\!\bigvee_{n=-\infty}^{\infty} (f = g^n)$), but "$C(g) \subseteq C(f)$" is a universal first-order formula, and thus holds in an existentially complete l-group G exactly when it holds in all extensions of G.

LEMMA 16. *If G is an existentially complete l-group and $e < f$, $g \in G$, then $f \in \langle g \rangle$ if and only if $G \models C(g) \subseteq C(f)$.*

PROOF. One way is obvious. For the other, assume $G \models C(g) \subseteq C(f)$ and assume $G \leqq B(T)$ for some o-2 homogeneous totally ordered set T. The assertion $f \in \langle g \rangle$ will follow from the two claims below:

(1) *For every $\alpha \in T$ there is a natural number $n(\alpha)$ such that $\alpha f = \alpha g^{n(\alpha)}$.* If not then $\alpha < \alpha f$ (otherwise set $n(\alpha) = 0$), and either $\alpha \leqq \alpha g^n < \alpha f$ for all n, or $\alpha g^n < \alpha f < \alpha g^{n+1}$ for some n. Suppose the former. If we set $\Gamma = \Delta(\alpha, f)$ then, in light of Lemma 15, either $\Gamma \cap \operatorname{supp}(g) = \varnothing$ or there is a supporting interval Δ of g such that $\Delta \subset (\beta, \beta f)$ for some $\beta \in \bar{\Gamma}$. In either case there is $y \in A(T)$ such that $yg = gy$ but $yf \neq fy$, a contradiction. (In the first case let y have support $\subseteq (\alpha, \alpha f)$. y is guaranteed to exist by the o-m transitivity and depressibility of $A(T)$. In the second case, let y be g depressed off Δ. If $\gamma \in \operatorname{supp}(y)$, then $\gamma f y = \gamma f < \gamma y f$.) Finally we suppose that for some n, $\alpha g^n < \alpha f < \alpha g^{n+1}$. By the o-m transitivity and depressibility of $A(T)$, there is $x_0 \in A(T)$ such that $\alpha f \in \operatorname{supp}(x_0) \subseteq (\alpha g^n, \alpha g^{n+1})$. Define $x \in A(T)$ so that $\operatorname{supp}(x) \subseteq \Gamma$ by setting $\beta x = \beta g^{n-m} x_0 g^{m-n}$ whenever $\beta \in [\alpha g^m, \alpha g^{m+1})$ for some $m \in \mathbf{Z}$, and $\beta x = \beta$ if $\beta \notin \bigcup_{m=-\infty}^{\infty} [\alpha g^m, \alpha g^{m+1})$. Clearly $xg = gx$. Since $\alpha g^n x_0 = \alpha g^n$, $\alpha g^m x = \alpha g^m$ for all $m \in \mathbf{Z}$. Hence $\alpha f = \alpha x f$. But $\alpha f x > \alpha f$ so $xf \neq fx$, a contradiction.

(2) *$n(\alpha)$ is independent of α.* Strictly speaking, $n(\alpha)$ is independent of α whenever $\alpha \in \operatorname{supp}(g)$; this n will of course work for any $\alpha \notin \operatorname{supp}(g)$ since, by (1), $\operatorname{supp}(f) \subseteq \operatorname{supp}(g)$. So assume $\alpha, \beta \in \operatorname{supp}(g)$. By the facts stated in the

introduction, there is an expansion of $A(T)$ to $A(T^*)$ and an element $h \in A(T^*)$ such that $\alpha h = \beta$ and $h^{-1}gh = g$. Thus $\beta f = \alpha h f = \alpha f h = \alpha g^{n(\alpha)}h = \alpha h g^{n(\alpha)} = \beta g^{n(\alpha)}$, so $n(\beta) = n(\alpha)$. This establishes Lemma 16.

Lemma 16 compares closely with a corresponding result for group theory. Angus Macintyre [12] proved that if G is an existentially complete group and $f, g_0, \cdots, g_{n-1} \in G$, then $f \in \langle g_0, \cdots, g_{n-1}\rangle$ if and only if $G \models \forall x\,[(\bigwedge_{i=0}^{n-1} xg_i = g_i x) \rightarrow xf = fx]$. We have no idea if Lemma 16 can be generalized even to capture $f \in \langle g_0, g_1\rangle$, or to the case that f, g aren't necessarily positive; e.g., is it possible for an existentially complete l-group to contain f, g such that $G \models \forall x\,(xg = gx \rightarrow xf = fx)$, f and g have the same supporting intervals and for each supporting interval Δ, $f|_\Delta = g|_\Delta$ or $f|_\Delta = g^{-1}|_\Delta$ (with both possibilities occurring as Δ runs through all the supporting intervals of g)? As indicated in the introductory remarks, Macintyre's proof is much shorter — and cleaner — relying heavily on free products with amalgamation.

THEOREM 17. *The theory of l-groups has no model companion.*

PROOF. This is cheap. Let G be any existentially complete l-group, and $\mathfrak{U}$ any non-principal ultrafilter on $\mathbf{Z}^+$. Let $e < g \in G$. Let $f, h \in \Pi G/\mathfrak{U}$ such that $f_n = g^n$ and $h_n = g$ $(n \in \mathbf{Z}^+)$. Now $\{n \in \mathbf{Z}^+ : C(h_n) \subseteq C(f_n)\} = \mathbf{Z}^+ \in \mathfrak{U}$ so $\Pi G/\mathfrak{U} \models \forall x\,(xh = hx \rightarrow xf = fx)$. However, $f \notin \langle h\rangle$, since $\mathfrak{U}$ is non-principal. By Lemma 15, $\Pi G/\mathfrak{U}$ is not existentially complete. Hence the theory of l-groups has no model companion.

THEOREM 18. *The theories of existentially complete l-groups and finitely generic l-groups are undecidable. Moreover, T^f is hyperarithmetic (as complicated as first order number theory).*

PROOF. Let G be existentially complete and $e < g \in G$. Using Lemma 15 we can code in first order arithmetic:

$$g^n \oplus g^m = g^p \leftrightarrow g^n \cdot g^m = g^p \quad (\text{i.e., } n + m = p) \qquad \text{and}$$

$$g^n \otimes g^m = g^p \leftrightarrow (\exists x)(x^{-1}gx = g^n \;\&\; x^{-1}g^m x = g^p) \quad (\text{i.e., } nm = p).$$

By Lemma 2, the model of arithmetic so obtained is independent of the choice of $g > e$. Thus, courtesy of K. Gödel, the theories are undecidable. By [10] theorem 7.7, T^f is hyperarithmetic.

This proof works equally well for groups, division rings, etc. (see [4], [10] and [13]).

On the one hand, Y. Gurevich has shown that the lattice theory of the theory of abelian lattice-ordered groups is undecidable [9]. On the other hand, as a consequence of Theorem 18:

THEOREM 19. *The group theory of the theory of l-groups is undecidable.*

PROOF. This is courtesy of A. Tarski [18]. Letting T be the group theory of the theory of l-groups, the previous proof shows that, in Tarski's terminology, relativized number theory is weakly interpretable in an inessential extension of T. His proof of undecidability of group theory now adapts to the undecidability of T.

To complete the proof of the Logic Theorem, we show that second order number theory can be interpreted in T^F, the infinite forcing companion (T enjoys the joint embedding property (since $G \oplus H$ is an l-group if G and H are), so T^F and T^f are complete ([7] appendix lemmas 11 and 21)).

Let $S(x_0, x_1)$ be the formula

$$(x_0 > e) \ \&\ (x_1 > e) \ \&\ (x_1^{-1}x_0x_1 \wedge x_0 = e).$$

LEMMA 20. *Let $X \subseteq \mathbf{Z}^+$. Then there exist $g_0, g_1 \in \mathscr{A}(\mathbf{R})$ such that $\mathscr{A}(\mathbf{R}) \models S(g_0, g_1^n)$ if and only if $n \in X$.*

PROOF. Enumerate $\mathbf{Z}^+ \backslash X$ in increasing order, say $m_0 < m_1 < m_2 < \cdots$. Let $e < g_0 \in \mathscr{A}(\mathbf{R})$ be such that $\operatorname{supp}(g_0) = \bigcup_{n=0}^{\infty}(2n, 2n+1)$. Define $e < g_1 \in \mathscr{A}(\mathbf{R})$ so that, for all $n \in \omega$, $(4n - \frac{1}{2})g_1 = 4n - \frac{1}{2}$, $(4n+2)g_1 > 4n+3$ and $(4n)g_1 > 4n+1$, and $g_1^{m_n} \mid (4n, 4n+1) = (4n+2, 4n+3)$. (If $\mathbf{Z}^+ \backslash X$ is finite so m_n doesn't exist, change the last clause to $(4n)g_1 \geqq 4n+3$.) Note that if $k \neq m_n$, then

$$g_1^{-k}g_0g_1^k \wedge g_0 \big|_{(4n,4(n+1))} = e \big|_{(4n,4(n+1))};$$

but

$$g_1^{-k}g_0g_1^k \wedge g_0 \big|_{(4n,4(n+1))} \neq e \big|_{(4n,4(n+1))} \qquad \text{if } k = m_n.$$

Hence $g_1^{-k}g_0g_1^k \wedge g_0 \neq e$ if and only if $k \in \mathbf{Z}^+ \backslash X$.

THEOREM 21. *T^F is Δ_1^2 (as complicated as second order number theory).*

PROOF. Let $G \geqq \mathscr{A}(\mathbf{R})$ be infinitely generic and $e < g \in G$. Given $X \subseteq \mathbf{Z}^+$, there exist $f_0, f_1 \in \mathscr{A}(\mathbf{R}) \leqq G$ such that $\mathscr{A}(\mathbf{R}) \models S(f_0, f_1^n)$ if and only if $n \in X$. But $e < f_1$, $g \in G$ so f_1 and g are conjugate in G (Lemma 2). Let $h \in G$ be such that $h^{-1}f_1h = g$. Since $g \models S(f_0, f_1^n)$ if and only if $n \in X$, $G \models S(h^{-1}f_0h, g^n)$ if and only if $n \in X$. Thus we can identify X with a definable subset of $\{g^n : n \in \mathbf{Z}^+\}$ (using $S(x_0, x_1)$ and $h^{-1}f_0h$). Thus we can code second order number theory into $Th(G)$

(see [10] chapter 16). But T^F is complete, so $T^F = Th(G)$. Hence T^F is Δ_1^2 by [10] theorem 7.6.

By Theorem 21 and [10] corollary 7.19:

COROLLARY 22. *There are $2^{\aleph_0}$ pairwise non-elementarily equivalent countable existentially complete l-groups (cf. Lemma 13).*

COROLLARY 23. *T^F and T^f share no models in common. Indeed, there is an $\forall\exists\forall$ sentence which distinguishes them.*

PROOF. See [10] chapter 16.

Note that an $\forall\exists\forall$ sentence is the simplest kind that could distinguish between the theories (see [7] appendix, proposition 22). Unfortunately, the $\forall\exists\forall$ sentence involves coding in Gödel numbers of formulae and proofs, and is not satisfying to an algebraist. Unfortunately, we have been unable to find a truly "algebraic" sentence (however complicated) to distinguish between T^F and T^f.

We now complete the proof of the Algebra Theorem by showing that existentially complete l-groups are never laterally complete. This fact is somewhat surprising since the non-existence of a supremum of some infinite subset of pairwise disjoint elements of an l-group has a decidedly second order flavor. We note that the techniques of [1] show that if H is an l-group, and $\{h_i : i \in I\}$ is an infinite set of pairwise disjoint elements having a supremum h in H, then there is an l-group $K \geqq H$ such that $\{h_i : i \in I\}$ has a supremum in K which isn't h.

THEOREM 24. *No existentially complete l-group is laterally complete.*

PROOF. If G is existentially complete, there exist positive g_0, g_1 in G such that $g_1^{-n} g_0 g_1^n \wedge g_0 = e$ for all $n \in \mathbf{Z}\backslash\{0\}$ (see the proof of Proposition 25). If G were laterally complete, $\bigvee_{n \in \mathbf{Z}} g_1^{-n} g_0 g_1^n$ would exist in G; call it f. Consider the sentence

$$\exists x\,(x \geqq g_1^{-1} g_0 g_1 \vee g_0 \vee g_1 g_0 g_1^{-1} \;\&\; x g_1 = g_1 x \;\&\; x < f).$$

Note that if $K \geqq G$ and $x = \bigvee_{n \in \mathbf{Z}} g_1^{-n} g_0 g_1^n$ in K, then $x \geqq g_1^{-1} g_0 g_1 \vee g_0 \vee g_1 g_0 g_1^{-1}$ and $x g_1 = g_1 x$. So if we take $K \geqq G$ an l-group in which $\bigvee_{n \in \mathbf{Z}} g_1^{-n} g_0 g_1^n$ exists and isn't f, the supremum in K satisfies the $\exists$ sentence. Hence

$$G \models \exists x\,(x \geqq g_1^{-1} g_0 g_1 \vee g_0 \vee g_1 g_0 g_1^{-1} \;\&\; x g_1 = g_1 x \;\&\; x < f).$$

Let $h \in G$ satisfy the formula. Then since $h g_1 = g_1 h$ and $h \geqq g_1^{-1} g_0 g_1 \vee g_1 g_0 g_1^{-1}$, $h \geqq g_1^{-n} g_0 g_1^n$ for all $n \in \mathbf{Z}$. Thus $f > h \geqq f = \bigvee_{n \in \mathbf{Z}} g_1^{-n} g_0 g_1^n$, a contradiction.

In a similar vein, we observe how unarchimedean existentially complete l-groups are:

PROPOSITION 25. *If G is existentially complete and $e < g \in G$, there exist positive $g_n \in G$ $(n \in \mathbf{Z})$ with $g_0 = g$ and $\cdots \ll g_{-2} \ll g_{-1} \ll g_0 \ll g_1 \ll g_2 \cdots$, where $x \ll y$ means $x^n \leqq y$ for all $n \in \mathbf{Z}^+$.*

PROOF. In $\mathcal{A}(\mathbf{R})$ we can find x, y with x positive and $x \wedge (y^{-1}xy)^{-1}x(y^{-1}xy) = e$. Hence $x \ll y^{-1}xy$. So in $G \oplus \mathcal{A}(\mathbf{R}) \models \exists x\, \exists y\, (x > e \;\&\; (y^{-1}xy)^{-1}x(y^{-1}xy) \wedge x = e)$ and hence the sentence holds in G. Let $x, y \in G$ be a solution. Since x and g are positive, they are conjugate (Lemma 2). Let $f^{-1}xf = g$. Then if $h = f^{-1}yf$, $(h^{-1}gh)^{-1}g(h^{-1}gh) \wedge g = e$ so $g \ll h^{-1}gh$. Letting $g_n = h^{-n}gh^n$ $(n \in \mathbf{Z})$ we obtain the lemma.

The reason we have not considered the theory of totally ordered groups and possible model companion is that we know almost nothing about existentially complete totally ordered groups. Algebraically closed totally ordered groups are simple (as totally ordered groups) and are either trivial or existentially complete by Bacsich's Theorem ([5] theorem 6.5 or [7], appendix theorem 3). We do not know if existentially complete totally ordered groups are divisible (it is a long-standing problem — due to B. H. Neumann — if every totally ordered group can be embedded in a divisible totally ordered group) or of any universal totally ordered groups to take the role that $\mathcal{A}(\Omega)$ played for l-groups.

Addendum. Let K be the group given by the presentation

$$(x_0, x_1 : x_0^{-1}x_1^2x_0 = x_1^{-2},\ x_1^{-1}x_0^2x_1 = x_0^{-2}).$$

Let $x_2 = x_0x_1$ and $u_i = x_i^2$ $(i = 0, 1, 2)$. Then K is torsion-free and cannot be right totally ordered (and hence cannot be lattice ordered). Moreover, A (the subgroup of K generated by u_0, u_1, u_2) is the free abelian group of rank 3 on u_0, u_1, u_2, and $A \lhd K$ (see Donald S. Passman, *The Algebraic Structure of Group Rings*, Wiley, 1977, p. 606). Let $N \lhd K$ with K/N torsion-free. If $N \neq \{e\}$, then $N \cap A \neq \{e\}$ since K/A has order 4. Let $e \neq y = u_0^m u_1^n u_2^p \in N$. Without loss of generality, $m \neq 0$. $z = u_0^{-2m}u_2^{-2p} = [y, x_1] \in N$ and hence $x_0^{8m} = u_0^{4m} = [z, x_2] \in N$. Therefore $x_0 \in N$. But $x_0x_1^4 = x_1^{-2}x_0x_1^2 \in N$ so $x_1 \in N$. Thus $N = K$ and K has no non-trivial torsion-free homomorphic images. Now let G be existentially complete in the class of torsion-free groups and ϕ be the sentence

$$\exists x_0\, \exists x_1\, (x_0^{-1}x_1^2x_0 = x_1^{-2} \;\&\; x_1^{-1}x_0^2x_1 = x_0^{-2} \;\&\; x_0 \neq e).$$

Then $G \times K \models \phi$ so $G \models \phi$. Hence G contains a subgroup isomorphic to K, and consequently G cannot be right totally ordered. It follows that:

Existentially complete lattice-ordered groups are not existentially complete in the class of torsion-free groups.

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